Transient Motion of a Hypersonic Wedge, Including Time History Effects

W.H. Hui* and H.J. Van Roessel†
University of Waterloo, Waterloo, Canada

The transient pitching motion of a wedge in hypersonic flow is studied by analytically solving the coupled dynamic/aerodynamic system. For small-amplitude motions, a perturbation method is used to derive a single functional/partial differential equation. It is found from the solution that the effects of past motion history of the body on the present state of its motion in hypersonic flow are entirely due to the wave reflection from the bow shock. These effects are shown to be important, causing a significant phase shift. In the limiting case of Newtonian flow, the perturbation theory is carried to the second order in amplitude, showing that the indicial response is a functional (rather than a function), depending on the whole past motion history.

I. Introduction

THE motion of an aircraft, treated here as a rigid body for simplicity, depends on the aerodynamic forces acting on it that, in turn, depend on the whole past history of the aircraft's motion. This coupling between the aircraft motion (described by the dynamic equations) and the airflow passing it (described by the Euler or Navier-Stokes equations) must be taken into account in order to correctly determine the maneuvering and dynamic stability of the aircraft.

In the past decades, Tobak has repeatedly pointed out the importance of studying the time history effects and he and his colleagues (see, e.g., Tobak and Schiff[1,2]) have made important advances in this area of research. The point was further emphasized in the recent lectures by Chapman and Tobak[3] and Hancock.[4] Tobak especially designed a mathematical model to approximate these effects of past history at various levels and his model has been verified to the second level (i.e., up to the immediate past) by Chyu and Schiff[5] using large-scale numerical computations.

At subsonic and low-supersonic speed, a general theory of unsteady lift accounting for the time-history effects was given by Lomax et al.[6] based on linear potential flow theory. They used an indicial-function representation of the aerodynamic response to incorporate the time-history effects. In doing so, they assumed the unsteady flow to be a small perturbation of some steady flow and linearized the equations. For low-supersonic speed, they also assumed the shock waves to be weak and replaced them with Mach waves. Their theory, based on the potential flow approach, cannot be easily extended to hypersonic flow where the shock waves are strong and can no longer be replaced by Mach waves.

The purpose of this paper is to study the time-history effects in hypersonic flow by considering the transient motion of a wedge. The shock wave behavior will be treated properly and special attention will be given to the wave reflection from the bow shock. Like Lomax et al.,[6] we consider the unsteady flow as a small perturbation to some steady flow and linearize the equations accordingly. Nonlinear effects will be considered for the limiting case of Newtonian flow only. The present study is restricted to that of a wedge, but otherwise constitutes a complementary study to Ref. 6 in that Ref. 6 treated time-history effects in low-supersonic and subsonic flow, whereas the present paper treats these effects in hypersonic flow.

II. Mathematical Formulation

Consider a wedge of semivertex angle σ placed symmetrically in a uniform supersonic/hypersonic stream M_∞. At time $t=0$, a disturbance is started and it is required to determine the subsequent pitching motion $\theta(t)$ of the wedge, due to the resulting unsteady aerodynamic forces, about a pivot axis located at a distance h from its apex along the centerline (Fig. 1). To fix the idea, we study only the case where the subsequent motion is a pitching motion in rectilinear flight. This corresponds to the wind tunnel experiment in which a uniform flow passes a model wedge that is allowed to perform pitching motion about the pivot axis.

The dynamic equation governing the motion of the rigid body is

$$\bar{I}\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = \bar{M}(t), \qquad t>0 \tag{1a}$$

$$\theta(0) = \tilde{\theta}_0 \tag{1b}$$

$$\dot{\theta}(0) = \tilde{\theta}_1 \tag{1c}$$

where $\theta(t)$ denotes the angular displacement of the motion from its steady flight position, $\bar{I}$ the moment of inertia of the body about the pivot axis, and $\bar{M}(t)$ the pitching moment (about the same axis) of the unsteady aerodynamic forces at time t. The latter are to be determined from the well-known equations of motion of an inviscid, perfect gas.

The pitching moment $\bar{M}(t)$ in Eq. (1a) is related to the surface pressure by

$$\bar{M}(t) = \iint_{B(x,t)=0} (x - x_c) \times p\frac{\nabla B}{|\nabla B|}\,\mathrm{d}S \tag{2}$$

where $B(x,t)=0$ denotes the body surface equation and x_c the position vector of the pivot axis in the plane $z=0$. The coupling of the wedge motion $\theta(t)$ with unsteady pressure p through the boundary condition [Eq. (4)] and the pitching moment [Eq. (2)] implies that $\bar{M}(t)$ depends not only on the present state of the motion $\theta(t)$ and $\dot{\theta}(t)$, but must depend on the whole past history of motion $\theta(\xi)$ $(0 \le \xi \le t)$. This means[2] that the pitching moment $\bar{M}$ is a functional of θ, i.e., $\bar{M}(t) = M[\theta(\xi)]$. To correctly determine the aircraft motion at any time $t>0$, it is therefore necessary to solve the coupled dynamic/aerodynamic system.

Submitted Jan. 10, 1985; presented as Paper 85-0201 at the AIAA 23rd Aerospace Sciences Meeting, Reno, NV, Jan. 14-17, 1985; revision received Sept. 3, 1985.

*Professor of Applied Mathematics and Mechanical Engineering. Member AIAA.

†Graduate student, Department of Applied Mathematics.

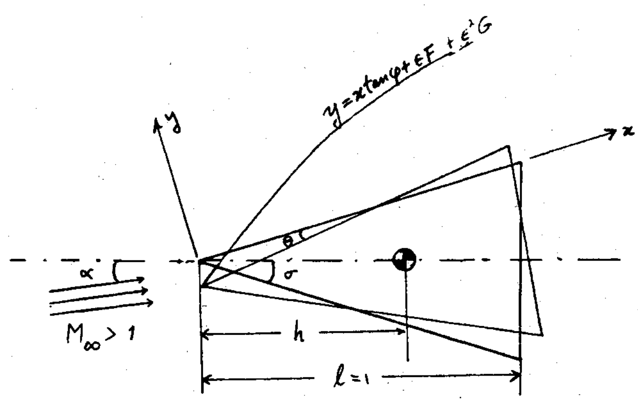

Fig. 1 Pitching wedge showing notation.

In this paper, all of the lengths are scaled by the length of the wedge ℓ, density by the freestream density ρ_∞, velocity components by the speed of freestream U_∞, pressure by $\rho_\infty U_\infty^2$, and the time variable by ℓ/u_0, where u_0 is given below.

Prior to time $t=0$, the flow over the wedge is the well-known steady flow past a wedge. Let the Cartesian system of coordinates $x0y$ be placed with the origin 0 at the apex and $0x$ along the wedge surface. Denote by u_0, a_0, p_0, and ρ_0, the flow speed, speed of sound, and pressure and density of the steady flow, respectively. Let the shock wave in the steady flow be $y = x\tan\phi$. These steady flow quantities are given in Appendix A.

After the motion is started, the equation of the surface is given by (see Fig. 1)

$$B(x,y,t) \equiv (x - \mathrm{h}\cos\sigma)\tan\theta(t) - y + \mathrm{h}\sin\sigma\frac{1-\cos\theta(t)}{\cos\theta(t)} = 0 \tag{3}$$

The boundary condition at the body surface [Eq. (3)] then becomes

$$u\tan\theta(t) - v + \frac{\dot\theta(t)}{\cos^2\theta(t)}[x - \mathrm{h}\cos\sigma + \mathrm{h}\sin\sigma\sin\theta(t)] = 0 \quad \text{at } B(x,y,t)=0 \tag{4}$$

where u and v are the x and y components of velocity, respectively.

Let the equation of the shock wave be

$$y - S(x,t) = 0 \tag{5}$$

then the Rankine-Hugoniot shock jump conditions become

$$u = \frac{2}{(\gamma+1)M_\infty^2}\frac{S_x}{Q} + \frac{u_\infty\left(1+\frac{\gamma-1}{\gamma+1}S_x^2\right)+\frac{2}{\gamma+1}S_x(v_\infty - S_t)}{1+S_x^2} \tag{6}$$

$$v = v_\infty + \frac{2}{\gamma+1}\left[\frac{Q}{1+S_x^2} - \frac{1}{M_\infty^2 Q}\right] \quad \text{at } y = S(x,t) \tag{7}$$

$$p = \frac{2Q^2}{(\gamma+1)(1+S_x^2)} - \frac{\gamma-1}{\gamma(\gamma+1)M_\infty^2} \tag{8}$$

$$\rho = \frac{Q^2}{[(\gamma-1)/(\gamma+1)]Q^2 + \{2/[(\gamma+1)M_\infty^2]\}(1+S_x^2)} \tag{9}$$

where

$$Q = S_t + u_\infty S_x - v_\infty \tag{10}$$

The position (x_A, y_A) of the apex of the wedge at time t is given by

$$x_A = h\{\cos\sigma - \cos[\sigma - \theta(t)]\} \tag{11a}$$

$$y_A = -h\{\sin\sigma - \sin[\sigma - \theta(t)]\} \tag{11b}$$

Hence the condition that the bow shock always attaches to the wedge apex requires

$$S(x,t) = -h\{\sin\sigma - \sin[\sigma-\theta(t)]\} \quad \text{at } x = h\{\cos\sigma - \cos[\sigma - \theta(t)]\} \tag{12}$$

To sum up, the mathematical problem of determining the transient motion of a wedge is that of solving the coupled system of Eqs. (1) and (2) and the equations of motion of an inviscid, perfect gas subject to the boundary conditions (4), (6-9), and (12). We shall assume the unsteady flow to be a small departure from some steady flow and use a perturbation method.

III. Perturbation Theory for Small Amplitude

Perturbation Equation

Let ϵ be a measure of the amplitude of the pitching motion of the body. We denote the angular displacement by $\epsilon\theta(t)$ and expand the flow variables as power series in ϵ, i.e.,

$$u = u_0(1 + \epsilon u_1 + \epsilon^2 u_2) + \mathcal{O}(\epsilon^3) \tag{13a}$$

$$v = u_0(\epsilon v_1 + \epsilon^2 v_2) + \mathcal{O}(\epsilon^3) \tag{13b}$$

$$p = p_0(1 + \epsilon p_1 + \epsilon^2 p_2) + \mathcal{O}(\epsilon^3) \tag{13c}$$

$$\rho = \rho_0(1 + \epsilon \rho_1 + \epsilon^2 \rho_2) + \mathcal{O}(\epsilon^3) \tag{13d}$$

and, for the shock wave,

$$S(x,t) = x\tan\phi + \epsilon F(x,t) + \epsilon^2 G(x,t) + \mathcal{O}(\epsilon^3) \tag{13e}$$

Substituting Eq. (13) into the equations of motion of an inviscid, perfect gas and boundary conditions (4-10) and (12) and equating like terms in ϵ, we get the following first-order problem for the determination of u_1, v_1, p_1, ρ_1, and F:

$$\frac{\partial\rho_1}{\partial t} + \frac{\partial\rho_1}{\partial x} + \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{14a}$$

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{1}{\gamma M_0^2}\frac{\partial p_1}{\partial x} = 0 \tag{14b}$$

$$\frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x} + \frac{1}{\gamma M_0^2}\frac{\partial p_1}{\partial y} = 0 \tag{14c}$$

$$\frac{\partial}{\partial t}(p_1 - \gamma\rho_1) + \frac{\partial}{\partial x}(p_1 - \gamma\rho_1) = 0 \tag{14d}$$

$$v_1 = \theta(t) + (x - h_1)\dot\theta(t), \quad \text{at } y = 0 \tag{15}$$

$$p_1 = A_1\frac{\partial F}{\partial t} + A_2\frac{\partial F}{\partial x} \tag{16a}$$

$$v_1 = B_1\frac{\partial F}{\partial t} + B_2\frac{\partial F}{\partial x}, \quad \text{at } y = x\tan\phi \tag{16b}$$

$$u_1 = C_1\frac{\partial F}{\partial t} + C_2\frac{\partial F}{\partial x} \tag{16c}$$

$$\rho_1 = D_1\frac{\partial F}{\partial t} + D_2\frac{\partial F}{\partial x} \tag{16d}$$

$$F(0,t) = -h_3\theta(t) \tag{17}$$

where $h_1 = h\cos\sigma$, $h_3 = h\cos\beta/\cos\phi$, and $\beta = \sigma + \phi$. The coefficients $A_1, \ldots, D_2$ in Eq. (16) are dependent on the steady flow and are given in Appendix B. They are valid for general wedge without using the hypersonic small-disturbance theory.

From Eqs. (14), a single wave equation for p_1 is obtained

$$\left(1 - \frac{1}{M_0^2}\right)\frac{\partial^2 p_1}{\partial x^2} + 2\frac{\partial^2 p_1}{\partial x\,\partial t} + \frac{\partial^2 p_1}{\partial t^2} - \frac{1}{M_0^2}\frac{\partial^2 p_1}{\partial y^2} = 0 \tag{18}$$

The First-Order Solution

To obtain analytic solutions, we further restrict ourselves to hypersonic flow past slender wedges. With $\sigma \ll 1$, $M_0^{-1} = \mathcal{O}(\sigma)$ and when $\mathcal{O}(\sigma^2)$ terms are neglected compared with unity, Eq. (18) reduces to

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^2 p_1 - \frac{1}{M_0^2}\frac{\partial^2 p_1}{\partial y^2} = 0 \tag{19}$$

The general solution to Eq. (19) is

$$p_1 = \gamma M_0[F_1(x - M_0 y, x - t) - F_2(x + M_0 y, x - t)] \tag{20}$$

with the corresponding solution for v_1 from Eq. (14c)

$$v_1 = F_1(x - M_0 y, x - t) + F_2(x + M_0 y, x - t) \tag{21}$$

Substituting Eqs. (20) and (21) into the boundary conditions [Eqs. (16a) and (16b)] at shock and solving for F_1 and F_2, we get

$$F_1[(1-H)x, x-t] = \frac{1}{2}\left[\left(B_1 + \frac{A_1}{\gamma M_0}\right)\frac{\partial F(x,t)}{\partial t} + \left(B_2 + \frac{A_2}{\gamma M_0}\right)\frac{\partial F(x,t)}{\partial x}\right] \tag{22}$$

$$F_2[(1+H)x, x-t] = \frac{1}{2}\left[\left(B_1 - \frac{A_1}{\gamma M_0}\right)\frac{\partial F(x,t)}{\partial t} + \left(B_2 - \frac{A_2}{\gamma M_0}\right)\frac{\partial F(x,t)}{\partial x}\right] \tag{23}$$

where

$$H = M_0 \tan\phi \tag{24}$$

Finally, substituting Eqs. (21-23) into the body surface boundary condition [Eq. (15)], we obtain the following functional partial differential equation for the determination of $F(\xi,\eta)$

$$\left.\frac{\partial F(\xi, t+Hx)}{\partial \xi}\right|_{\xi=x} + (1+\mu)\left.\frac{\partial F(\xi, t+Hx)}{\partial t}\right|_{\xi=x} - \lambda\left.\frac{\partial F(\xi, t-mHx)}{\partial \xi}\right|_{\xi=mx} - (\lambda-\nu)\left.\frac{\partial F(\xi, t-mHx)}{\partial t}\right|_{\xi=mx} = \frac{2}{a}[\theta(t) + \{(1-H)x - h_1\}\dot\theta(t)] \tag{25}$$

where

$$m = \frac{1-H}{1+H}, \quad a = \frac{A_2}{\gamma M_0} + B_2, \quad \lambda = \left(\frac{A_2}{\gamma M_0} - B_2\right)\Big/ a \tag{26}$$

$$\mu = \left(B_1 - B_2 + \frac{A_1 - A_2}{\gamma M_0}\right)\Big/ a, \quad \nu = \left(B_1 - B_2 - \frac{A_1 - A_2}{\gamma M_0}\right)\Big/ a$$

Equation (25) is the generalization of the functional differential equation derived by Hui [Ref. 7, Eq. (5)] in studying the interaction of Mach waves with strong shock in unsteady flow. The constants m, a, λ, μ, ν are the same as in Ref. 7. It can be shown as in Ref. 7 that λ is the reflection coefficient of a pressure wave equal to the ratio of the amplitude of the pressure disturbance reflected from the bow shock to that incident on the bow shock.

With the hypersonic small-disturbance approximation, Eqs. (26) simplifies to[7]

$$H = \left[\frac{2 + (\gamma-1)K^2}{2\gamma K^2 - (\gamma-1)}\right]^{\frac{1}{2}}, \quad a = \frac{2[1 + (1+2H)K^2]}{(\gamma+1)K^2}$$
$$\lambda = \frac{(2H-1)K^2 - 1}{(2H+1)K^2 + 1}, \quad \mu = \nu = 0 \tag{26a}$$

where $K = M_\infty \beta$ is the hypersonic similarity parameter based on shock angle β. Consequently, Eq. (25) reduces to

$$\frac{\partial F(\xi,\eta)}{\partial \xi} + \frac{\partial F(\xi,\eta)}{\partial \eta} - \lambda\left[\frac{\partial F(m\xi, \eta-\xi+m\xi)}{\partial \xi} + \frac{\partial F(m\xi, \eta-\xi+m\xi)}{\partial \eta}\right] = \frac{2}{a}\{\theta(\eta - H\xi) + [(1-H)\xi - h]\dot\theta(\eta - H\xi)\} \tag{27}$$

The shock attachment condition [Eq. (17)] also simplifies to

$$F(0,t) = -h\theta(t) \tag{28}$$

The solution to Eq. (27) that satisfies Eq. (28) is

$$F(\xi,\eta) = \frac{2}{ab_0}\left[(b_0\xi - h)\theta(\eta - H\xi) + \left(\frac{m}{m-\lambda} - \frac{ab_0}{2}\right)h\theta(\eta - \xi) + \sum_{n=1}^{\infty}\left(\frac{\lambda}{m}\right)^n (b_n\xi - h)\theta(\eta - \xi + b_n\xi)\right] \tag{29}$$

where

$$b_n = (1-H)m^n, \quad n = 0,1,2,\ldots \tag{30}$$

With the first-order shock shape function $F(x,t)$ completely determined by the angular displacement $\theta(t)$ through Eq. (29), the first-order flow quantities p_1 and v_1 can be obtained from Eqs. (20) and (21), u_1 and ρ_1 may also be obtained easily by solving Eq. (14b) to satisfy Eq. (16c) and solving Eq. (14d) to satisfy Eq. (16d), respectively. The first-order unsteady pitching moment $M(t)$ of the wedge at time t about the pivot axis is twice the contribution from the upper surface and is given by

$$\frac{\bar M}{\rho_\infty U_\infty^2 \ell^2} = M(t) = -2\epsilon p_0 \int_0^1 (x-h)\left.p_1\right|_{y=0} \mathrm{d}x \tag{31}$$

Now, use the solution of Eq. (29) for $F(x,t)$ to get F_1 and F_2 from Eqs. (22) and (23), then p_1 from Eq. (20), and finally

$M(t)$ from Eq. (31). Thus,

$$\frac{M(t)}{\rho_0 u_0 a_0} = -(1-2h)\theta(t) - \frac{2}{3}(1-3h+3h^2)\dot{\theta}(t)$$
$$-4\sum_{n=1}^{\infty}\frac{\lambda^n}{t_n}\left[h^2\theta(t)-(1-h)(1-h-t_n)\theta(t-t_n)\right]$$
$$+8h\sum_{n=1}^{\infty}\frac{\lambda^n}{t_n^2}\int_0^{t_n}\theta(t-\tau)\,\mathrm{d}\tau$$
$$-4\sum_{n=1}^{\infty}\frac{(2-t_n)\lambda^n}{t_n^3}\int_0^{t_n}\tau\theta(t-\tau)\,\mathrm{d}\tau \tag{32a}$$

where

$$t_n = 1 - m^n \tag{32b}$$

The system governing the transient pitching motion $\theta(t)$ is thus given by

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = M(t) \tag{33a}$$

$$\theta(0) = \tilde{\theta}_0 \tag{33b}$$

$$\dot{\theta}(0) = \tilde{\theta}_1 \tag{33c}$$

where

$$I = \frac{\bar{I}}{\rho_\infty (U_\infty/u_0)^2 \ell^4} \tag{33d}$$

As seen from the derivations given above, the system of Eqs. (33), subject to linear approximation, fully incorporates the past motion history and the interaction between the body motion and the aerodynamic forces in determining the motion state of the body at the present time. We note that, due to the presence of the integral terms in Eq. (32a) for $M(t)$, to determine a solution of Eq. (33a) we need, in addition to the initial data [Eqs. (33b) (33c)], to specify the past motion history $f(t)$, i.e.,

$$\theta(t) = f(t) \qquad (-1 < t < 0) \tag{33e}$$

Physical Mechanism Causing Time History Effects

We now discuss the pitching moment $M(t)$ at time t. As seen from Eqs. (32), it contains three different types of terms. Thus, the first two terms on the right-hand side (RHS) depend only on the instantaneous state of the motion $\theta(t)$ and $\dot{\theta}(t)$ at time t. The third term on the RHS depends not only on the value of θ at the instantaneous time t, but also on its values at discrete previous times $t_1, t_2, t_3, \ldots$; this term arises because of wave reflection from the bow shock and was already obtainable[7,8] by assuming the motion of the wedge to be harmonic. Finally, the last two terms on the RHS involving integrals are new results, representing additional time history effects depending on the whole past motion history. They are also due to wave reflection at the bow shock.

If the effects of reflected waves are neglected, i.e., $\lambda = 0$, Eq. (32a) reduces to

$$M(t) = -2\rho_0 u_0 a_0\left[\left(\tfrac{1}{2}-h\right)\theta(t) + \left(\tfrac{1}{3}-h+h^2\right)\dot{\theta}(t)\right] \tag{34}$$

It is known[7] that the coefficients $(\frac{1}{2}-h)$ and $(\frac{1}{3}-h+h^2)$ are indeed the stiffness and damping-in-pitch derivatives of the oscillating wedge when wave reflection is ignored. The effects of wave reflection are justifiably neglected in Ref. 6 as they are generally insignificant for the weak shock waves encountered at low supersonic Mach number. However, they become important in hypersonic flow when the shock is strong and can no longer be neglected.[7,8]

Equation (32a) clearly shows in the hypersonic flow case that the effects of past motion history of the wedge on its pitching moment at present time are entirely due to the wave reflection from the bow shock. Since time is measured by ℓ/U_∞, one unit of time is that required by a fluid particle to travel from the apex to the end of the wedge. Hence, $t_n = 1 - m^n$ represents the time required for a particle to travel from the point P_n, whose coordinate is $x = m^n$, to the point P at the end of the wedge $x = 1$. It is clear from Fig. 2 that a signal originating at P_n will be reflected n times at the bow shock before it reaches P. Upon each such reflection, the signal is weakened by a factor λ. The term $\lambda^n\theta(t-t_n)$ and the integral term $\lambda^n\int_0^{t_n}\theta(t-\tau)\,\mathrm{d}\tau$ in Eq. (32a) evidently represent the contribution to the pitching moment $M(t)$ of the wedge motion during the interval from past time $(t-t_n)$ to the present time t.

As noted in the introduction, the work of Lomax et al.[6] on unsteady lift using potential theory includes time history effects in low-supersonic flow, while the present study aims at the same effects in hypersonic flow. The physical mechanisms causing time history effects appear quite different in the hypersonic flow case than in the low-supersonic flow case. Due to the inherent hypersonic assumption, it is not possible to compare directly the present result [Eq. (32a)] with Ref. 6 for low M_∞ flow. However, it is worth pointing out that according to Ref. 6 the history effects tend to vanish as M_∞ increases, while the present study shows that the wave reflection mechanism required to move the shock wave to a new position after a step change in θ extends the influence of the transient moment by a significant amount and the history effects do not vanish with increasing M_∞.

The Transient Motion

The solution to Eq. (33), obtained by the method of Laplace transform, is

$$\theta(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} T(s)\,e^{st}\,\mathrm{d}s \tag{35a}$$

where the Laplace transform $T(s)$ of $\theta(t)$ is

$$T(s) = \frac{(2\kappa+s)\tilde{\theta}_0 + \tilde{\theta}_1 + A(s)}{s^2 + 2\kappa s + (1-2h)/N - B(s)} \tag{35b}$$

with

$$N = I/(\rho_0 u_0 a_0), \quad \kappa = \left(\tfrac{1}{3}-h+h^2\right)/N$$

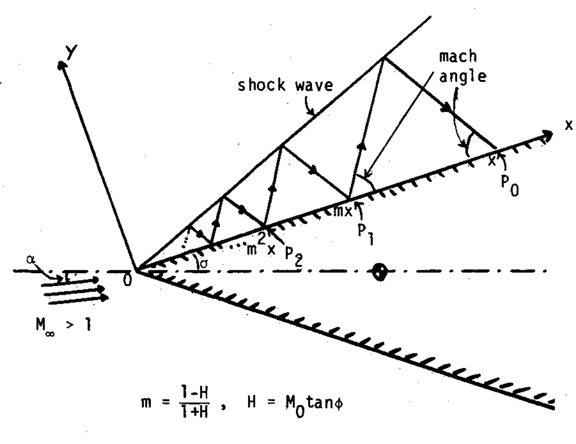

Fig. 2 Wave reflection.

$$A(s)=\frac{4}{N}\sum_{n=1}^{\infty}\frac{\lambda^n}{t_n s^2}\{[\gamma_n s^2-(\alpha_n-\beta_n t_n)s+\beta_n]e^{-t_n s}F_n(s)$$
$$+(\alpha_n s-\beta_n)C_{1n}+\beta_n C_{2n}s\}$$
$$B(s)=\frac{4}{N}\sum_{n=1}^{\infty}\frac{\lambda^n}{t_n s^2}\{[\gamma_n s^2-(\alpha_n-\beta_n t_n)s+\beta_n]e^{-t_n s}$$
$$-h^2s^2+\alpha_n s-\beta_n\}$$
$$\alpha_n=2h/t_n,\ \beta_n=(1+m^n)/t_n^2,\ \gamma_n=(1-h)(m^n-h)$$
$$C_{1n}=\int_{-t_n}^{0}f(\tau)\,\mathrm{d}\tau,\ C_{2n}=\int_{-t_n}^{0}\tau f(\tau)\,\mathrm{d}\tau$$
$$F_n(s)=\int_{-t_n}^{0}f(\tau)e^{-s\tau}\,\mathrm{d}\tau$$

For the special case where the density of the wedge is a constant ρ_w, we get from Eq. (33d)

$$I=(2\rho_w/\rho_\infty)\sin\sigma\cos\sigma\left(\tfrac{1}{3}-h+h^2+\tan^2\sigma\right)\tag{36}$$

Under the hypersonic small disturbance approximation, $\rho_0 u_0 a_0=\beta/H$; hence

$$N=\frac{4H}{\gamma+1}\frac{\rho_w}{\rho_\infty}\frac{K^2-1}{K^2}\left(\frac{1}{3}-h+h^2\right)\tag{37a}$$

$$\kappa=\frac{\gamma+1}{4}\frac{\rho_\infty}{\rho_w}\frac{K^2}{H(K^2-1)}\tag{37b}$$

Traditionally, the motion of an airfoil is determined from the inertial equation

$$2I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=(-C_{m_\theta})\theta(t)+(-C_{m_{\dot\theta}})\dot\theta(t)\tag{38a}$$
$$\theta(0)=\tilde\theta_0\tag{38b}$$
$$\dot\theta(0)=\tilde\theta_1\tag{38c}$$

where the stiffness derivative $(-C_{m_\theta})$ and the damping-in-pitch derivative $(-C_{m_{\dot\theta}})$ are given by Eq. (34) without wave reflection or are given by Refs. 7 and 8 with wave reflection. Neither case fully includes the time history effects.

In obtaining the solution of Eq. (33), we use the solution of Eq. (38) for $-1<t<0$ as $f(t)$ in Eq. (33e). In other words, the two motions are identical up to $t=0$, but after $t=0$ the traditional solution of Eq. (38) does not fully include time history effects, whereas Eq. (34) does. An example showing the time history effects is given in Fig. 3. It is seen from the figure that the wave reflection is very important: it causes a significant phase shift.

IV. Finite Amplitude Motion in Newtonian Flow

Second-Order Solution

The perturbation scheme of Sec. III can, in principle, be carried to second and higher orders to determine the effects of finite amplitude on the transient motion. To avoid complicated algebra, we shall limit ourselves to the Newtonian flow case, but place no restriction on the thickness σ of the wedge. This is done by taking the limit $M_\infty\to\infty$ and $\gamma\to 1$ of the gasdynamic theory in Sec. III.

For a wedge of semivertex angle σ placed at an angle of attack $\alpha<\sigma$ (Fig. 1) in a Newtonian flow, the body surface pressure is given in the form

$$p(\xi,t)=p_0(1+\epsilon p_1+\epsilon^2 p_2)+\mathcal{O}(\epsilon^3)\tag{39}$$

where[9]

$$p_0=\sin^2\psi\tag{40a}$$

$$p_0p_1(\xi,t)=j\sin2\psi[\theta(t)+(2\xi-h_1)\dot\theta(t)+\tfrac{1}{2}\xi(\xi-h_1)\ddot\theta(t)]\tag{40b}$$

$$\begin{aligned}p_0p_2(\xi,t)&=\theta^2(t)\cos2\psi+\theta(t)\dot\theta(t)(4\xi+2h_1m_1)\\&+\theta(t)\ddot\theta(t)(\xi-h_1)(m_2\xi-h_1m_3)\\&+\dot\theta^2(t)\cos^2\psi\left[(\xi-h_1)^2+\xi h_1\tan\sigma\tan\psi\right]\\&+[2\dot\theta(t)+(\xi-h_1)\ddot\theta(t)][\xi\tan^2\psi+h_1m_4]\\&\times\cos^2\psi\,\theta(t-\xi\sec\psi)\\&-[2\dot\theta(t)\sin^2\psi\cos\psi+\ddot\theta(t)(1+\cos\psi)\\&\times\cos^2\psi(\xi-h_1)]\cdot\int_0^{\xi\sec\psi}\theta(t-\tau)\,\mathrm{d}\tau\end{aligned}\tag{40c}$$

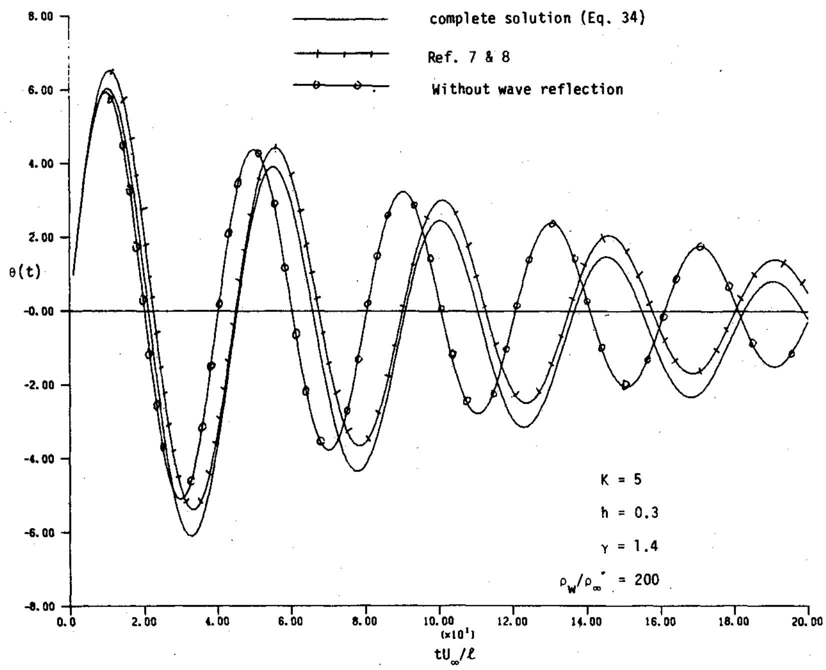

Fig. 3 Transient motion of a slender wedge.

where

$$m_1 = \cos\psi(\sin\sigma\sin\psi - 2\cos\sigma\cos\psi)/\cos\sigma$$
$$m_2 = 2 + \cos\psi$$
$$m_3 = \cos\psi\cos(\sigma+\psi)/\cos\sigma$$
$$m_4 = 1 - \tan\sigma\tan\psi \qquad (40d)$$

In Eq. (39), ξ is the coordinate along the instantaneous body surface from its apex, and

$$\psi = \sigma + j\alpha \qquad (41)$$

with $j=-1$ denoting the upper surface, while $j=1$ the lower surface. It is noted that, apart from a difference in notation, Eq. (40b) includes as a special case the result of Eq. (37) of Ref. 10 when the motion is assumed harmonic.

We observe from Eq. (40b) that at zero angle of attack $\alpha=0$ and $p_1|_{\text{upper surface}} = -p_1|_{\text{lower surface}}$, both contributing equal amounts to the pitching moment M. On the other hand, from Eq. (40c) we get $p_2|_{\text{upper surface}} = p_2|_{\text{lower surface}}$, implying that the second-order pitching moment vanishes at $\alpha=0$. Such a second-order pitching moment is, however, not zero if $\alpha \neq 0$, as is easily seen from Eq. (40c).

The pitching moment from the upper and lower surface is given by

$$M(t) = \sum_{j=-1}^{1} j\int_0^{\sec\sigma} -(\xi-h_1)\,p(\xi,t)\,d\xi = \epsilon M_1(t) + \epsilon^2 M_2(t) + \mathcal{O}(\epsilon^3) \qquad (42)$$

where from Eq. (40b)

$$M_1(t) = M_1[\theta(t)] = -\sum_{j=-1}^{1} j\int_0^{\sec\sigma}(\xi-h_1)\,p_0 p_1(\xi,t)\,d\xi$$
$$= -4\tan\sigma\cos2\alpha\left(\frac{1}{2}-h_2\right)\theta(t) - \frac{2\tan\sigma}{\cos\sigma}\cos\alpha\left(\frac{4}{3}-3h_2+2h_2^2\right)\dot\theta(t) - \frac{\tan(\sigma+\alpha)+\tan(\sigma-\alpha)}{\cos^2\sigma}\left(\frac{1}{4}-\frac{2}{3}h_2+\frac{1}{2}h_2^2\right)\ddot\theta(t) \qquad (43a)$$

with $h_2 = h\cos^2\sigma$. Similarly, from Eq. (40c) we get

$$M_2(t) = M_2[\theta(t)] = -\sum_{j=-1}^{1} j\int_0^{\sec\sigma}(\xi-h_1)\,p_0 p_2(\xi,t)\,d\xi$$
$$= -\sum_{j=-1}^{1} j\left\{\frac{\theta^2(t)\cos2\psi}{\cos^2\sigma}\left(\frac{1}{2}-h_2\right) + \frac{\theta(t)\dot\theta(t)}{\cos^3\sigma}\left[\frac{4}{3}+(m_1-2)h_2-2m_1h_2^2\right] + \frac{\theta(t)\ddot\theta(t)}{\cos^4\sigma}\left[\frac{m_2}{4}-\frac{1}{3}(2m_2+m_3)h_2 + (m_2+2m_3)h_2^2 - m_3h_2^3\right] + \frac{\dot\theta^2(t)\cos^2\psi}{\cos^4\sigma}\times\left[\frac{1}{4}-\frac{1}{3}(2+m_4)h_2+\frac{1}{2}(2+m_4)h_2^2-h_2^3\right]\right.$$
$$+\cos^2\psi\int_0^{\sec\sigma}\{\xi^3\ddot\theta(t)\tan^2\psi + \xi^2[\ddot\theta(t)h_1m_4 - 2\ddot\theta(t)h_1\tan^2\psi + 2\dot\theta(t)\tan^2\psi] + \xi[-2\ddot\theta(t)h_1^2m_4 + 2\dot\theta(t)h_1m_4 - 2\dot\theta(t)h_1\tan^2\psi] + [\ddot\theta(t)h_1 - 2\dot\theta(t)]h_1^2m_4\}$$
$$\times\theta(t-\xi\sec\psi)\,d\xi$$
$$\left. - \int_0^{\sec\sigma}\left[\ddot\theta(t)(\xi-h_1)^2(1+\cos\psi)\cos^2\psi + 2\dot\theta(t)(\xi-h_1)\sin^2\psi\cos\psi\right]\int_0^{\xi\sec}\theta(t-\tau)\,d\tau\,d\xi\right\} \qquad (43b)$$

Substitution of Eqs. (42) and (43) into Eqs. (33a) yields

$$2(I+I_a)\ddot\theta(t) + D\dot\theta(t) + S\theta(t) = 2\epsilon M_2[\theta(t)] \qquad (44)$$

where

$$D = \frac{4\tan\sigma}{\cos\sigma}\cos\alpha\left(\frac{4}{3}-3h_2+2h_2^2\right) \qquad (45a)$$

$$S = 4\tan\sigma\cos2\alpha(1-2h_2) \qquad (45b)$$

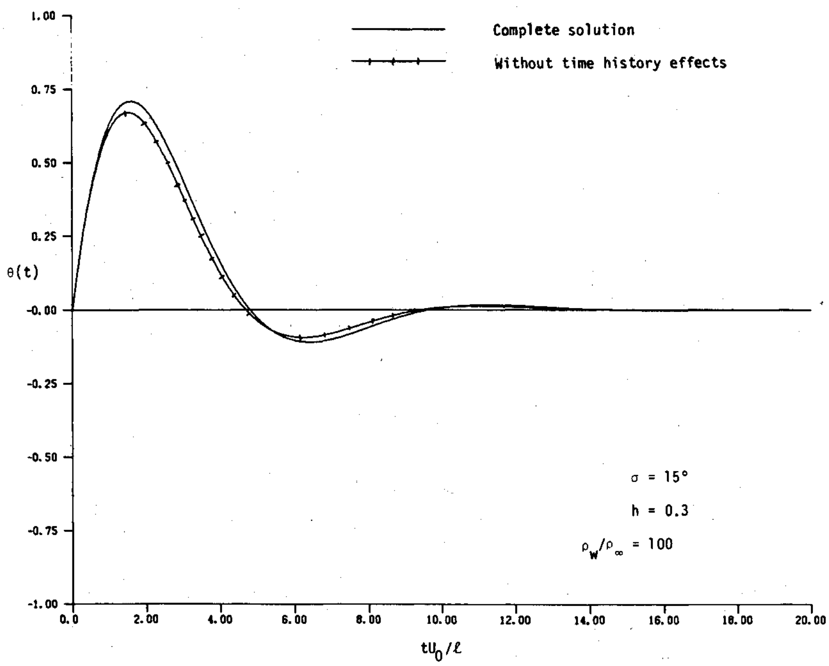

Fig. 4 Transient motion of a wedge in Newtonian flow.

D and S are the damping-in-pitch derivative and the stiffness derivative, respectively. Equations (45) include the results of Eqs. (61) and (62) in Ref. 12 as special case when the angle of attack $\alpha = 0$. The added moment of inertia I_a in Eq. (44) is

$$I_a = \frac{\tan(\sigma+\alpha)+\tan(\sigma-\alpha)}{\cos^2\sigma}\left(\frac{1}{4}-\frac{2}{3}h_2+\frac{1}{2}h_2^2\right) \quad (46)$$

which is always positive.

When the second-order terms (in amplitude ϵ) are neglected, the RHS of Eq. (44) vanishes. We thus see that to the first order in the Newtonian flow, the time history effects are equivalent to adding a moment of inertia I_a. This has the effect of reducing the rate of damping of oscillation from $D/2I$ to $D/2(I+I_a)$, making the oscillatory motion more persistent. It also modifies the frequency of oscillation slightly. The effects of time history depend clearly on the ratio I_a/I. Using Eqs. (36) and (46) at $\alpha = 0$, this ratio is

$$\frac{I_a}{I} = \frac{\rho_\infty}{\rho_w}\frac{1}{\cos^2\sigma}\frac{\frac{1}{4}-\frac{2}{3}h\cos^2\sigma+\frac{1}{2}h^2\cos^4\sigma}{\left(\frac{1}{3}-h+h^2\right)\cos^2\sigma+\sin^2\sigma} \quad (47)$$

An example of the time history effects is given in Fig. 4.

Indicial Response

Since the unsteady pressure and pitching moment at time t are obtained fully and explicitly as functionals of the motion history, e.g.,

$$M(t) = M\left[\begin{matrix}\tau=t\\ \theta(\tau)\\ \tau=0\end{matrix}\right]$$

in Eq. (32) or (39), we may use them to study the properties of the corresponding indicial responses. From the structure of Eqs. (32) and (40), it is easily shown that for small-amplitude motions the indicial response of the pitching moment at time t will be a function (rather than a functional) of $\theta(t)$. This corresponds to the second-level approximation in Refs. 2 and 13. However, for large-amplitude motions, due to the nonlinear terms in Eq. (40c), the indicial response at time t will be a functional, not a function, of $\theta(\tau)$. These conclusions are in agreement with Ref. 13 and are expected to hold true for finite Mach number cases as well.

Conclusions

An analytic method is developed for calculating the transient pitching motion of a wedge in hypersonic flow, taking into account fully the interaction between its motion and the unsteady airflow passing it. The effects of past motion history on the present state of motion are shown to arise due to the wave reflection from the bow shock. They are shown to be important, causing significant phase shift.

In the Newtonian limit the results are greatly simplified. In particular, to the first order in amplitude the time history effects are equivalent to an added moment of inertia. Finally, finite amplitude of motion causes the indicial response itself to become a functional rather than just a function as in the small-amplitude linear theory.

Appendix A: Steady Flow over a Wedge

Given the freestream Mach number $M_\infty > 1$, the ratio of specific heats of the gas γ, and the semiwedge angle σ, the steady uniform flow over the wedge is given below. The shock angle β is the middle root of the cubic equation for $\tan\beta$

$$\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\tan^3\beta - \left(M_\infty^2-1\right)\cot\sigma\tan^2\beta + \left(1+\frac{\gamma+1}{2}M_\infty^2\right)\tan\beta + \cot\sigma = 0 \quad (A1)$$

The Mach number M_0, pressure p_0, density ρ_0, and velocity u_0 are then given by

$$p_0 = \frac{2}{\gamma(\gamma+1)M_\infty^2}\left[\gamma M_\infty^2\sin^2\beta - \frac{\gamma-1}{2}\right] \quad (A2)$$

$$M_0^2\sin^2\phi = \frac{1+\frac{\gamma-1}{2}M_\infty^2\sin^2\beta}{\gamma M_\infty^2\sin^2\beta-\frac{\gamma-1}{2}} \quad (A3)$$

$$\rho_0 = \frac{(\gamma+1)M_\infty^2\sin^2\beta}{(\gamma-1)M_\infty^2\sin^2\beta+2} \quad (A4)$$

$$u_0 = a_0 M_0 \quad (A5)$$

where

$$\phi = \beta - \sigma \quad (A6)$$

$$a_0^2 = \gamma p_0/\rho_0 \quad (A7)$$

Let

$$\delta = \frac{\gamma-1}{\gamma+1} + \frac{1}{M_\infty^2\sin^2\sigma} \quad (A8)$$

In the Newtonian limit as $M_\infty \to \infty$ and $\gamma \to 1$, we expand as a power series in δ to get

$$\phi = \delta\phi_1, \quad \frac{1}{M_\infty^2} = \delta\mu_1, \quad \gamma = 1+\delta\gamma_1 \quad (A9)$$

then Eqs. (A1-A5) yield

$$\phi_1 = \frac{\mu_1 + (\gamma_1/2)\sin^2\sigma}{\sin\sigma\cos\sigma} \quad (A10)$$

$$M_0^2 = \frac{1}{\delta\phi_1\tan\sigma} + \mathcal{O}(1) \quad (A11)$$

$$p_0 = \sin^2\sigma + \mathcal{O}(\delta) \quad (A12)$$

$$\rho_0 = \frac{\tan\sigma}{\delta\phi_1} + \mathcal{O}(1) \quad (A13)$$

$$u_0 = \cos\sigma + \mathcal{O}(\delta) \quad (A14)$$

Appendix B

The coefficients $A_1, A_2, \ldots, D_2$ in Eq. (16) are

$$A_1 = 2\gamma M_0^2\tilde{K}\left[1-\frac{\gamma-1}{2}\tilde{W}(\rho_0-1)\right]\tan\phi\cos^2\phi$$

$$A_2 = \cos^2\phi A_1$$

$$B_1 = \tilde{K}\left[1+\rho_0\tilde{W}-\gamma\tilde{W}(\rho_0-1)\right]\cos^2\phi$$

$$B_2 = \tilde{K}\left[1+\rho_0\left(M_0^2-1\right)\tan^2\phi-\gamma\tilde{W}(\rho_0-1)\right]\cos^4\phi$$

$$C_1 = -B_1\tan\phi$$

$$C_2 = (B_2-B_1)\cot\phi$$

$$D_1 = M_0^2\tilde{K}\left[(\gamma+1)-(\gamma-1)\rho_0\right]\tan\phi\cos^2\phi$$

$$D_2 = D_1\cos^2\phi$$

where

$$\tilde{W} = M_0^2\sin^2\phi$$

$$\tilde{K} = \frac{1-(1/\rho_0)}{(1-\tilde{W})}$$

In the Newtonian limit, from results in Appendix A, we get

$$\tilde{W} \to \delta\phi_1/\tan\sigma \to 0$$

$$\rho_0 \tilde{W} \to 1$$

$$\rho_0 M_0^2 \tan^2\phi \to 1$$

$$A_1 = A_2 = 2\cot\sigma$$

$$B_1 = B_2 = 1$$

$$C_1 = 0, \qquad C_2 = -\tan\sigma$$

$$D_1 = D_2 = 2\cot\sigma - \gamma_1/\phi_1$$

Acknowledgments

This research was supported by a NASA Grant NAGW-575 and the Natural Science and Engineering Research Council of Canada. We are grateful to M. Tobak and G.T. Chapman of NASA Ames Research Center for their valuable comments on the work.

References

[1] Tobak, M. and Schiff, L.B., "The Role of Time-History Effects in the Formulation of the Aerodynamics of Aircraft Dynamics," *Dynamic Stability Parameters*, AGARD CP-235, May, 1978, Paper 26.

[2] Tobak, M. and Schiff, L.B., "Aerodynamic Mathematical Modelling—Basic Concepts," AGARD LS-114. 1981, Paper 1.

[3] Chapman, G.T. and Tobak, M., "Nonliner Problems in Fluid Dynamics," *Proceedings of Workshop on Nonlinear Problems in Control and Fluid Dynamics*, edited by L.R. Hunt and C.F. Martin, Mathematics and Science Press, 1984, pp. 1-56.

[4] Hancock, G.J., "Role of Unsteady Aerodynamics in Aircraft Response," AGARD R-679, 1980, Paper 8.

[5] Chyu, W.J. and Schiff, L.B., "Nonlinear Aerodynamic Modeling of Flap Oscillations in Transonic Flow: A Numerical Validation," *AIAA Journal*, Vol. 21, 1983, pp. 106-113.

[6] Lomax, H., Heaslet, M.A., Fuller, F.B., and Shuder, L., "Two- and Three-Dimensional Unsteady Lift Problems in High Speed Flight," NACA Rept. 1077, 1950.

[7] Hui, W.H., "Interaction of a Strong Shock with Mach Waves in Unsteady Flow," *AIAA Journal*, Vol. 7, 1969, pp. 1605-1607.

[8] McIntosh, S.C. Jr., "Hypersonic Flow over an Oscillating Wedge," *AIAA Journal*, Vol. 3, 1965, pp. 433-440.

[9] Van Roessel, H.J., Ph.D. thesis, University of Waterloo, Waterloo, Canada, 1985.

[10] Hui, W.H., "Unsteady Newton-Busemann Flow Theory, Part III: Frequency Dependence and Indicial Response," *The Aeronautical Quarterly*, Vol. 33, 1983, pp. 311-328.

[11] Tobak, M., "On the Use of the Indicial Response Function Concept in the Analysis of Unsteady Motions of Wings and Wing-Tail Combinations," NACA Rept. 1184, 1954.

[12] Hui, W.H. and Tobak, M., "Unsteady Newton-Busemann Flow Theory, Part I: Aerofoils," *AIAA Journal*, Vol. 19, 1981, pp. 311-318 (see also *Applied Mechanics Reviews*, Vol. 35, 1982, p. 1787; Letters to Editors, p. 12103).

[13] Tobak, M., Chapman, G.T., and Schiff, L.B., "Mathematical Modeling of the Aerodynamic Characteristics in Flight Dynamics," *Proceedings of Workshop on Nonlinear Problems in Control and Fluid Dynamics*, edited by L.R. Hunt and C.F. Martin, Mathematics and Science Press, 1984, pp. 435-450.